接続係数 (クリストッフェル記号) の変換

接続係数は次のように変換します：

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial x^{\beta'}}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\Gamma^{\alpha}{}_{\beta\gamma} + \frac{\partial x^{\alpha'}}{\partial x^{\sigma}}\frac{\partial^2 x^{\sigma}}{\partial x^{\beta'}\partial x^{\gamma'}} \tag{1}$$

Proof. $\boldsymbol{e}_\beta$ を基底ベクトルとするとき，接続係数 $\Gamma^{\alpha}{}_{\beta\gamma}$ は次を満たすものとして定義されました：

$$\frac{\partial \boldsymbol{e}_\beta}{\partial x^\gamma} = \Gamma^{\alpha}{}_{\beta\gamma}\boldsymbol{e}_\alpha \tag{2}$$

この定義より，もう一つの座標系をプライム『$'$』付き座標系とすれば,

$$\frac{\partial \boldsymbol{e}_{\beta'}}{\partial x^{\gamma'}} = \Gamma^{\alpha'}{}_{\beta'\gamma'}\boldsymbol{e}_{\alpha'} = \Gamma^{\alpha'}{}_{\beta'\gamma'}\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\boldsymbol{e}_\alpha \tag{3}$$

が成り立ちます．一方，この式の左辺は次のように計算することができます：

$$\begin{aligned}
\frac{\partial \boldsymbol{e}_{\beta'}}{\partial x^{\gamma'}} =& \frac{\partial}{\partial x^{\gamma'}}\left(\frac{\partial x^{\beta}}{\partial x^{\beta'}}\boldsymbol{e}_\beta\right) \\
=& \frac{\partial^2 x^{\beta}}{\partial x^{\gamma'}\partial x^{\beta'}}\boldsymbol{e}_\beta + \frac{\partial x^{\beta}}{\partial x^{\beta'}}\frac{\partial \boldsymbol{e}_\beta}{\partial x^{\gamma'}} \\
=& \frac{\partial^2 x^{\beta}}{\partial x^{\gamma'}\partial x^{\beta'}}\boldsymbol{e}_\beta + \frac{\partial x^{\beta}}{\partial x^{\beta'}}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\frac{\partial \boldsymbol{e}_\beta}{\partial x^{\gamma}} \\
=& \frac{\partial^2 x^{\beta}}{\partial x^{\gamma'}\partial x^{\beta'}}\boldsymbol{e}_\beta + \frac{\partial x^{\beta}}{\partial x^{\beta'}}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\Gamma^{\alpha}{}_{\beta\gamma}\boldsymbol{e}_\alpha \\
=& \frac{\partial^2 x^{\alpha}}{\partial x^{\gamma'}\partial x^{\beta'}}\boldsymbol{e}_\alpha + \frac{\partial x^{\beta}}{\partial x^{\beta'}}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\Gamma^{\alpha}{}_{\beta\gamma}\boldsymbol{e}_\alpha \\
=& \left(\frac{\partial^2 x^{\alpha}}{\partial x^{\gamma'}\partial x^{\beta'}} + \frac{\partial x^{\beta}}{\partial x^{\beta'}}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\Gamma^{\alpha}{}_{\beta\gamma}\right)\boldsymbol{e}_\alpha
\end{aligned} \tag{4}$$

ここで (3)=(4) ですから,

$$\Gamma^{\alpha'}{}_{\beta'\gamma'}\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\boldsymbol{e}_\alpha = \left(\frac{\partial^2 x^{\alpha}}{\partial x^{\gamma'}\partial x^{\beta'}} + \frac{\partial x^{\beta}}{\partial x^{\beta'}}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\Gamma^{\alpha}{}_{\beta\gamma}\right)\boldsymbol{e}_\alpha$$

が成り立ちますので，基底 $\boldsymbol{e}_\alpha$ が互いに線形独立であることより，各係数が一致しなくてはなりません．したがって，

$$\Gamma^{\alpha'}{}_{\beta'\gamma'}\frac{\partial x^\alpha}{\partial x^{\alpha'}} = \frac{\partial^2 x^\alpha}{\partial x^{\gamma'}\partial x^{\beta'}} + \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}}\Gamma^\alpha{}_{\beta\gamma}$$

が成り立ちます．すると，この式の両辺に $\frac{\partial x^{\delta'}}{\partial x^\alpha}$ を掛けて，縮約をとると，左辺は,

$$\Gamma^{\alpha'}{}_{\beta'\gamma'}\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^{\delta'}}{\partial x^\alpha} = \Gamma^{\alpha'}{}_{\beta'\gamma'}\frac{\partial x^{\delta'}}{\partial x^{\alpha'}} = \Gamma^{\alpha'}{}_{\beta'\gamma'}\delta^{\delta'}_{\alpha'} = \Gamma^{\delta'}{}_{\beta'\gamma'}$$

となります．一方右辺は,

$$\frac{\partial x^{\delta'}}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x^{\gamma'}\partial x^{\beta'}} + \frac{\partial x^{\delta'}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}}\Gamma^\alpha{}_{\beta\gamma}$$

となりますので，両辺を等号で結ぶと結局,

$$\Gamma^{\delta'}{}_{\beta'\gamma'} = \frac{\partial x^{\delta'}}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x^{\gamma'}\partial x^{\beta'}} + \frac{\partial x^{\delta'}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}}\Gamma^\alpha{}_{\beta\gamma}$$

が成り立つことになります．両辺の添字の δ' を α' に変え，ダミー添字を変えて整理すると，最終的に接続係数の変換は,

接続係数の変換

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}}\Gamma^\alpha{}_{\beta\gamma} + \frac{\partial x^{\alpha'}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^{\beta'}\partial x^{\gamma'}}$$

となることが分かりました．これを見れば分かる通り，接続係数はテンソルとして変換しないことがわかります[*1]．

□

[*1] ※このようなものを擬テンソルと呼びます．